Is a Scientific Idea of Hope Possible Dr. Andrew W. Harrell

John Calvin and the Dutch Bishop Jansen believed that Hope comes about against all odds. But, the French Theologian Blaise Pascal was trying to defend Jansen against criticism from the Jesuits and the Church that they were not consistent with St. Augustine's teachings. To do this he had to try and explain what "Hope with the odds" might mean. Along the way the foundations were laid for our modern understanding of what mathematical "probability" means. Do you agree with how Pascal defines probability? And what type of Hope do you believe in (against all odds or with the odds). The talk in this MAS section will concentrate on the details of how Pascal laid the foundations for modern axiomatic probability theory in order to solve the problem in gambling of a "game of points". Along the way he proved some fascinating theorems on the properties of the binomial coefficients that occur in his "arithmetic triangle" and are used today to create combinatorial statistical testing designs for experiments, fractal computer images, and much more.

- How did Pascal proved his key theorem that justifies the use of his arithmetical triangle to calculate probabilistic “expectations” [hopes] for the gamblers problem in the “game of points”. The solution to this problem was an important event in 17th century mathematics and allowed Pascal to create a new mathematical discipline called probability theory. The mathematican Blaise Pascal spent a good part of his life thinking about the scientific/theological question of how to define probability

- **The game of points studied by Fermat and Pascal consisted of a stake being made by both players, then a fair coin being flipped a number of times.**
- The stake goes to whichever player has a given number of heads or the most heads at the end of the flipping. If, at a certain point in the game [before the given number of points has been achieved by either player] they decide to quit and divide up winnings so far, the question comes up, “how is this to be done?”. Thus the game is an example of what is called nowadays a Bernoulli trial [the probability of the results of each flip are independent of the results that have gone before]. It is clear from the fact that their are only two possibilities for each throw that a given game of n points can only last for at most 2n throws. And, if, at a certain point in game it m out of 2n throws have already occurred, then in order to figure out the solution to the problem as stated, it is only necessary to consider the possibilities and combinations of throws out to 2*(n-m) more throws.
- After Fermat’s letter [see Great Books volume 33 reference] Pascal started thinking about if the game was analyzed during it course of play how to we define the probability for one player winning, considered what has happened up to that time.
- **In order to do this he formulated a tentative definition, later to be made more precise by Laplace:** “The theory of chance consists in reducing all the events of the same kind to a certain number of cases equally possible, that is to say, to such as we may be equally undecided about in regard to their existence, and in determining the number of cases favorable to the event whose probability is sought. The ratio of this number to that of all the cases possible is the measure of this probability, which is thus simply a fraction whose numerator is the number of favorable cases and whose denominator is the number of all the cases possible.” — Pierre-Simon Laplace,
- A Philosophical Essay on Probabilities

- Certain rules were postulated around this time by mathematicians studying in this area (or assumed) as to how to calculate "probabilities p(E) of an event E"
- 1) for all events $0 < p(E) < 1$
- 2) p(impossible events) = 0. p(certain events)=1
- 3) p(not an event happening)= 1- p(E)
- 4) If two events, A,B are disjoint in occurence: p(A or B)= p(A) + p(B)
- 5) If two events A,B are with independently determined outcomes, but But successive results of the same experiments p(A then B) =p(A and B)= p(A)*p(B) In order to determine the ratio of favorable possible outcomes to unfavorable at a particular point in the analysis of the partial results of the complete results

- **Pascal computed a lot of the values of what we now call, due to Newton I believe, binomial coefficients**[because they are the coefficients of the expansion of the powers a binomial function (a + b)^n [(a+b) raised to the power n] = sum of terms k=1 to n (n k) a^k*b^(n-k) (. If you think about it in terms of how the powers of a^k*b^(n-l) you will understand that these coefficients count the number of ways of picking k objects out of a set of n. In the course of doing this he discovered the important step by step relationship which allows us to compute a table of the values of these coefficients for a given n, assuming we already know what the coefficients are for the case of n-1.
- Pascal's algorithm to compute his famous arithmetic triangle which can be used to compute binomial coefficients First of all, by the definition of the binomial coefficients in terms of combinations of k objects picked out of n we know they have the property that the number is (k n)=n* (n-1)..(n-k+1)/k*(k-1)…1

- In Pascal's paper he wrote his triangle rotated 45 degrees. But, in order to understand If in relation to the binomial coefficients lets start out with it in the form
- 1
- 1 1
- 1 2 1
- 1 3 3 1
- 1 4 6 4 1
- It is defined inductively using the relation between the coefficients (which is easy to verify from by writing out the above definition and getting a common denominator):
- (k n) = (k n-1) + (k -1 n-1)
- (k n)=n* (n-1)..(n-k+1)/k*(k*(k-1)..1= (n-1)*(n-2)…(n-k+1/k*(k-1)*(k-2)…1 +
- (n-1)*(n-2)..(n-k)/(k-1)*(k-2)..1
- Thus, for example, this relation says that the 2nd coefficient in the third row (2 3)
- Is the sum of the 2nd coefficient in the 2nd row (2 2) and the 1st coefficient in the 2nd row (1 2).
- Or writing it out:
- 3*2/2*1=6/2=3=2*1/2 +1/1=2+1=3

- **The formula can be proved algebraically, as above, using the definition of the binomial coefficients in terms of factorials.**
- **Or, a simple combinatorial argument using the definition in terms of k subsets of n objects, can be given:**
- Of the set of n objects, let one be called T. All of the k subsets are either of two types:
- a) those that contain T (which number (k-1 n-1))
- b) those that do not (which number (k n-1)
- thus (k n) = (k-1 n-1) + (k n-1) Q.E.D.

- For the case n =6 referred to in his letter to Fermat [triangle is rotated
- 45 degrees here]
- cells numbered along the top are called numbered by parallel rows [PLR]
- cells numbered from top to bottom are called numbered by perpendicular rows
- [PPR]
- and cells numbered from the top left to the bottom center are called numbered
- by the base [BR]
- Thus, the algorithm says the for any cell (PLR=n PPR= k), its value [the
- value in the cell]
- AT(k n) = AT(k n-1)[the value in cell on its left] + AT(k-1 n)[the value just
- on top of it]
- 1 2 3 4 5 6
- 1 1 1 1 1 1 1
- 2 1 2 3 4 5
- 3 1 3 6 10
- 4 1 4 10
- 5 1 5
- 6 1

In this display the row and column indices surround the arithmetic triangle itself

- **In his essay Pascal after defining the triangle and its parameters he proves some simple theorems about the triangle thus rotated:**
- 1) Each cell is equal to the sum of all the cells of the preceding row from its own perpendicular row to the first, inclusive. Eg. the value at cell 3,3 = 6 which equals the values at cells 2,1 +2,2 + 2,3 = 1+ 2+ 6
- 2) In each such triangle the value at each cell exceeds by unity the sum of all the cells within its parallel
- And perpendicular rows, exclusive. Eg. the value at cell 3,3=6 which equals the values at cells
- 3,1+3,2 +1,2+2,2 = 1+3+1+2=7
- 3) each cell is equal to its reciprocal eg. 3,2=3 is equal to 2,3=3
- 4) the sum of all the cells of each base is double that of the preceding base. Eg.
- sum of cells of base 3 1+3+3+1=8 ½ sum of cells of base 4 = 1+ 4+6+4+1=16
- 5) If we compare the sum of the coefficients [(0 j) +(1 j) + (2 j) +... (j j)] = X (in the triangle to their interpretation as polynomial coefficients in the term by term expansion of (1+1)^j we find the fact that X = 2^j.
- 6) If, on the other hand, we look at the polynomial expansion of (1 -1)^j, we find the fact that
- [(0 j) - (1 j) + (2 j) - ... (-1)^j (j j)] = 0

- **Those interested can consult the excellent Mathematical Association of America's (MAA) introductory text "Mathematics of Choice, How to Count without Counting" by Dr. Ivan Niven** and the more advanced one "Combinatorial Mathematics" by Herbert Ryser for more details on these Theorems . Also, the MAA's book, "The Mathematics of Games and Gambling" by Edward Packel has a lot of information about how to use binomial coefficients to compute the probabilities of results in backgammon, roulette and craps and also bridge and poker hands. Binomial coefficients and the arithmetic triangle have many applications. In addition to the theory of games and gambling already mentioned, then can be used to count out and construct symmetrical experimental designs, patterns of symmetry and tiling's in continuous and discrete transformation groups, fractal image compression transformations (see my UC Berkeley graduate mathematics department friend Hans Otto Pietgen and his several Springer books on Fractals).

- Another one of the applications that Pascal gives of the arithmetrical triangle is to compute what he defines as the "expected payoff" or "probabilistic expectation" or "mathematical expectation" at a given position in a game of one player versus the other in the game of points.
- Let E1, E2,E3,E4...EN be pairwise disjoint events (no pair can occur simulataneously) with their respective probabilities P1,P2,P3,P4...PN. Then what we call the "mathematical expectation" in an experiment in which one of these events must occur is defined to be equal to:
- P1*E1+P2*E2+P3*E3+P4*E4 +...PN*EN
- **Pascal's Theorem** [used to compute expectation values for the game of points]:
- In a game of points
- in which each player lacks a certain number of points r[1st player] and s [2nd player] to win:
- In order to find the division of expectations [hopes for a win of the stakes]
- Use the arithmetical triangle with base of order n=r+s and add up the coefficients of the base of
- The triangle of that order in that proportion [the sum of those r coefficients is compared
- To the sum of those s]
- The sum of all the coefficients in the base row is equal to the sum of the binomial coefficients of
- the ways k objects can be taken from n[computed by means of the formula to be
- equal to (k n)= n*(n-1)…(n-k+1)/k*(k-1)…1) where n is the number of the base
- And k is the index running diagonally along the base.

- In order to prove the validity of the formula for the expectations of each player:
- expectation of 1st player = number of ways of winning in last r+s throws/ total number of possible throws
- expectation of 2nd player = number of ways of winning in last r+s throws/total number of possible throws
- we use the 4th axiom listed above which define the probability of a series of disjoint events {the throws} being the sum of the probability of each. And, we also use the 3rd axiom to tell us that to get the probability at each throw we multiply the independent particular combinations of probability of separate dice outcomes times the total number of possible outcomes.
- In order to prove the validity of the algorithm for all triangles [and hence all game situations], since we have already identified the expectation as a ratio of sums of binomial coefficients and we know the base rows in the triangle are made up of these coefficients, defined inductively and recursively by Pascal's formula (which already has been proved in two ways) the theorem follows by induction on the base index of the triangle, by using base index as the inductive index.

- How is this theorem of Pascal used to compute the answer to the earlier question
- About stopping a game of stakes midway in the game?
- In the game of stakes for n heads to win, with one player needing k more
- and the other n-k, you can sum up the 1st k columns and compare it to the sum
- of the last n-k columns and you have their respective "hopes with the odds "
- for a final win.
- For example, if two players have wagered money on being able to win three
- times, and the 1st player has already won twice and the second once. If two
- more throws happen there are four possibilities,
- 1) the 1st player wins both
- 2) the 1st player wins the next one and the second player the last one
- 3) the 2nd player wins the next one and the 1st the last one
- 4) the 2nd player wins both
- In the first three cases the 1st player wins the game and in the fourth case
- the 2nd player does.

- By the accepted reasoning up to that time if you quit before the throws you
- would divide up the wagers with ¾ to the 1st player and ¼ to the 2nd
- But, according to Pascal's "method of expectations." we need to take into
- account all the possible ways these things can happen in two different extra
- throws. Computing the triangle we have:
- 1 1 1
- 1 2
- 1
- . The 1st can win in 1 +2 ways according to Pascal and the 2nd in 1 way. Thus
- the odds are 3 to 1. But, according to the argument of Cardano it would be 3
- to 4.
- By mathematical induction, and using the recursive properties of the binomial
- coefficients (n k)(which can also be computed, interpreting them as
- combinations as n*(n-1)…(n-k+1)/k*(k-1)…1)
- triangle, Pascal was able to solve the problem for any number of points the
- players might
- lack.

- In the case of 5 points to win, stopped when the 1st player has 3 points and the second 2 points
- 1 2 3 4 5
- 1 1 1 1 1 1
- 2 1 2 3 4
- 3 1 3 6
- 4 1 4
- 5 1
- the odds will be (1+4+6)=11 to (4+1)= 5 in the favor of the 1st player and so the money should be given to him in the ratio of 11/5.
- And, in the case of 6 throws which corresponds to the 1st triangle listed
- above, if the 1st player lacks four points and the 2nd lacks two, then their
- shares are found by adding the numbers in the base of the triangle in Figure
- 1 2 3 4 5 6
- 1 1 1 1 1 1 1
- 2 1 2 3 4 5
- 3 1 3 6 10
- 4 1 4 10
- 5 1 5
- 6 1
- 1: the 1st
- share is to the 2nd as 1+5+10+10 to 5+1, or 13 to 3.

- Thank you for coming. For those interested in a discussion of how Pascal’s thoughts apply to theological problems (like whether it makes rational sense to believe in God) I am giving a talk tomorrow morning in the History and Philosophy of Science Division.